

VOLTERRA TYPE OPERATORS ON GROWTH FOCK SPACES

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ABSTRACT. Let ω be an unbounded radial weight on \mathbb{C}^d , $d \geq 1$. Using results related to approximation of ω by entire maps, we investigate Volterra type and weighted composition operators defined on the growth space $\mathcal{A}^\omega(\mathbb{C}^d)$. Special attention is given to the operators defined on the growth Fock spaces.

1. INTRODUCTION

Let $\mathcal{H}ol(\mathbb{C}^d)$ denote the space of entire functions on \mathbb{C}^d , $d \geq 1$. Given a symbol $g \in \mathcal{H}ol(\mathbb{C}^d)$, the extended Cesàro operator V_g is defined as

$$V_g f(z) = \int_0^1 f(tz) \mathcal{R}g(tz) \frac{dt}{t},$$

where $f \in \mathcal{H}ol(\mathbb{C}^d)$, $z \in \mathbb{C}^d$ and $\mathcal{R}g(z) = \sum_{j=1}^d z_j \frac{\partial g}{\partial z_j}(z)$ is the radial derivative of g . Relations between the function theoretic properties of the symbol g and the operator theoretic properties of V_g have been intensively investigated after the works of Pommerenke [13], and Aleman and Siskakis [2] for $d = 1$. Given a holomorphic map $\varphi : \mathbb{C}^d \rightarrow \mathbb{C}^d$, we also consider the composition operator $C_\varphi : f \mapsto f \circ \varphi$. The superpositions $V_g^\varphi = V_g \circ C_\varphi$ and $C_{\varphi,g} = C_\varphi \circ V_g$ are called Volterra type operators.

1.1. Fock spaces. The present paper is motivated by recent studies of V_g , V_g^φ , $C_{\varphi,g}$ and related operators defined on or mapping to the Fock type spaces $\mathcal{F}_\alpha^p(\mathbb{C}^d)$, $\alpha > 0$, $0 < p \leq \infty$; see, for example, [6, 9, 10, 11, 12, 14, 15]. The weighted Fock space $\mathcal{F}_\alpha^p(\mathbb{C}^d)$, $d \geq 1$, $0 < p < \infty$, $\alpha > 0$, consists of those $f \in \mathcal{H}ol(\mathbb{C}^d)$ for which

$$\|f\|_{\mathcal{F}_\alpha^p}^p = \left(\frac{\alpha p}{2\pi}\right)^d \int_{\mathbb{C}^d} |f(z)|^p e^{-\frac{\alpha p}{2}|z|^2} d\nu_d(z) < \infty,$$

where ν_d denotes Lebesgue measure on \mathbb{C}^d . For $p = \infty$, the corresponding growth Fock space $\mathcal{F}_\alpha^\infty(\mathbb{C}^d)$ consists of those $f \in \mathcal{H}ol(\mathbb{C}^d)$ for which

$$\|f\|_{\mathcal{F}_\alpha^\infty} = \sup_{z \in \mathbb{C}^d} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty.$$

The operators $V_g^\varphi : \mathcal{F}_\alpha^p(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ with $p = \infty$ or $q = \infty$ were investigated by Mengestie [10]. In fact, the case $q = \infty$ fits into certain general schemes. For example, given $g \in \mathcal{H}ol(\mathbb{C}^d)$, consider the weighted composition operator $C_\varphi^g f(z) = g(z) C_\varphi f(z)$, $z \in \mathbb{C}^d$. Given a Banach space $X \subset \mathcal{H}ol(\mathbb{C}^d)$, the bounded operators $C_\varphi^g : X \rightarrow \mathcal{F}_\alpha^\infty(\mathbb{C}^d)$ are characterizable in terms of α, d and the norms $\|\delta_z\|_{X^*}$, $z \in \mathbb{C}^d$, where $\delta_z(f) = f(z)$. In particular, for the unit disk \mathbb{D} of \mathbb{C} and $X \subset \mathcal{H}ol(\mathbb{D})$,

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similar schemes were used in [7] and [5] for $C_\varphi^g : X \rightarrow Y$, where Y is a growth, weighted Bloch or Lipschitz space of arbitrary order. The corresponding compact operators are described by related little-oh conditions. So, we concentrate our attention on the case where $p = \infty$.

For $p = \infty$, as in the work of Ueki [14], the characterizations of bounded (compact) operators V_g^φ are based in [10] on Berezin type integral transforms; see Section 4 for details. We use approximation of radial weights by appropriate entire functions to obtain more explicit descriptions of the bounded (compact) V_g^φ and similar operators on $\mathcal{F}_\alpha^\infty(\mathbb{C}^d)$, $d \geq 1$, and on general growth spaces $\mathcal{A}^\omega(\mathbb{C}^d)$, $d \geq 1$.

1.2. Growth spaces. Let $\omega : [0, +\infty) \rightarrow (0, +\infty)$ be a weight function, that is, let ω be non-decreasing, continuous and unbounded. Setting $\omega(z) = \omega(|z|)$ for $z \in \mathbb{C}^d$, we extend ω to a radial weight on \mathbb{C}^d . The corresponding growth space $\mathcal{A}^\omega(\mathbb{C}^d)$ consists of those $f \in \mathcal{H}ol(\mathbb{C}^d)$ for which

$$\|f\|_{\mathcal{A}^\omega} = \sup_{z \in \mathbb{C}^d} \frac{|f(z)|}{\omega(z)} < \infty.$$

Given a set X and functions $u, v : X \rightarrow (0, +\infty)$, we write $u \asymp v$ and we say that u and v are equivalent if $C_1 u(x) \leq v(x) \leq C_2 u(x)$, $x \in X$, for some constants $C_1, C_2 > 0$. Replacing ω in the definition of $\mathcal{A}^\omega(\mathbb{C}^d)$ by an equivalent weight function, we clearly obtain the same space, with an equivalent norm. In what follows, we also always assume that $\lim_{t \rightarrow +\infty} t^{-k} \omega(t) = \infty$ for all $k \in \mathbb{N}$. This condition excludes finite-dimensional spaces $\mathcal{A}^\omega(\mathbb{C}^d)$ of polynomial growth.

1.2.1. Associated weight functions. To work with arbitrary weight functions, we use the notion of associated weight formally introduced in [3].

Definition 1.1. Let $v : [0, +\infty) \rightarrow (0, +\infty)$ be a weight function. For $d \geq 1$, let $v_d(z) = v(|z|)$, $z \in \mathbb{C}^d$, be the corresponding radial weight on \mathbb{C}^d . The associated weight \tilde{v}_d is defined by

$$\tilde{v}_d(z) = \sup\{|f(z)| : f \in \mathcal{H}ol(\mathbb{C}^d), |f| \leq v_d \text{ on } \mathbb{C}^d\}, \quad z \in \mathbb{C}^d.$$

As observed in [3], \tilde{v}_1 is a radial weight, hence, the associated weight function $\tilde{v}_1 : [0, +\infty) \rightarrow (0, +\infty)$ is uniquely defined. Using compositions with unitary transformations of \mathbb{C}^d , we conclude that \tilde{v}_d is radial for all $d \geq 1$. Observe that a different notion of radial weight is used in [3] for $d \geq 2$. Also, standard arguments guarantee that the identity $\tilde{v}_d = \tilde{v} := \tilde{v}_1$, $d \geq 2$, holds for the associated weight functions.

1.2.2. Essential weight functions. Let $\omega : [0, +\infty) \rightarrow (0, +\infty)$ be an arbitrary weight function. The weight function $\tilde{\omega}$ correctly defines the corresponding associated weight on \mathbb{C}^d for any $d \geq 1$. Also, the definition of the associated weight guarantees that $\mathcal{A}^\omega(\mathbb{C}^d) = \mathcal{A}^{\tilde{\omega}}(\mathbb{C}^d)$ isometrically. However, working with $\mathcal{A}^{\tilde{\omega}}(\mathbb{C}^d)$, we obtain answers in terms of $\tilde{\omega}$. So, as in [3], we say that ω is essential if ω is equivalent to $\tilde{\omega}$ on $[0, +\infty)$. Also, it is known and easy to see that $(\tilde{\omega}) = \tilde{\omega}$, hence, $\tilde{\omega}$ is always essential. This observation allows to apply approximation results from [1] to arbitrary weight functions.

1.3. Results. In this section, we formulate several typical results of the present paper. Our first theorem is a general fact related to operators mapping into lattices, spaces with minimal structure assumptions. Let $\mathcal{L}(\mathbb{C}^d)$ be a linear space whose elements are functions $F : \mathbb{C}^d \rightarrow \mathbb{C}$. We say that $\mathcal{L}(\mathbb{C}^d)$ is a lattice if the following property holds: Assume that $f, F : \mathbb{C}^d \rightarrow \mathbb{C}$ are continuous functions and $|f(z)| \leq |F(z)|$, $z \in \mathbb{C}^d$. If $F \in \mathcal{L}(\mathbb{C}^d)$, then $f \in \mathcal{L}(\mathbb{C}^d)$.

Theorem 1.2. Suppose that $\omega : [0, +\infty) \rightarrow (0, +\infty)$ is a weight function, $g \in \mathcal{H}ol(\mathbb{C}^d)$, $d \geq 1$, $\varphi : \mathbb{C}^d \rightarrow \mathbb{C}^d$ is a holomorphic map, and $\mathcal{L}(\mathbb{C}^d)$ is a lattice. Then the weighted composition operator C_φ^g maps $\mathcal{A}^\omega(\mathbb{C}^d)$ into $\mathcal{L}(\mathbb{C}^d)$ if and only if

$$(1.1) \quad |g(z)|\tilde{\omega}(|\varphi(z)|) \in \mathcal{L}(\mathbb{C}^d),$$

where $\tilde{\omega}$ denotes the associated weight function.

In the present paper, $\mathcal{L}(\mathbb{C}^d)$ is usually a weighted L^q space. In particular, for Volterra type operators mapping into Fock type spaces, we have the following corollaries.

Corollary 1.3. Let ω be an arbitrary radial weight on \mathbb{C}^d , $d \geq 1$. Assume that $g \in \mathcal{H}ol(\mathbb{C}^d)$, $\varphi : \mathbb{C}^d \rightarrow \mathbb{C}^d$ is a holomorphic map, $\alpha > 0$ and $0 < q \leq \infty$. Then the following properties are equivalent:

- (i) $V_g^\varphi : \mathcal{A}^\omega(\mathbb{C}^d) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C}^d)$ is a bounded operator;
- (ii)

$$\frac{|\mathcal{R}g(z)|}{(1+|z|)^2} e^{-\frac{\alpha}{2}|z|^2} \tilde{\omega}(|\varphi(z)|) \in L^q(\mathbb{C}^d, \nu_d).$$

Corollary 1.4. Let ω be a radial weight on \mathbb{C} . Assume that $\varphi, g \in \mathcal{H}ol(\mathbb{C})$, $\alpha > 0$ and $0 < q \leq \infty$. Then the following properties are equivalent:

- (i) $C_{\varphi, g} : \mathcal{A}^\omega(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ is a bounded operator;
- (ii)

$$\frac{|g'(z)\varphi'(z)|}{1+|z|} e^{-\frac{\alpha}{2}|z|^2} \tilde{\omega}(|\varphi(z)|) \in L^q(\mathbb{C}, \nu_1).$$

Corollary 1.4 and analogous results are restricted to spaces of holomorphic functions on \mathbb{C} , since the corresponding proofs depend on differentiation of compositions.

Also, we obtain similar explicit results for T_g , the companion to V_g defined as

$$T_g f(z) = \int_0^1 \mathcal{R}f(tz)g(tz) \frac{dt}{t}, \quad f \in \mathcal{H}ol(\mathbb{C}^d), \quad z \in \mathbb{C}^d.$$

Berezin type characterizations of such operators were obtained in [12].

1.4. Organization of the paper. Auxiliary facts are collected in Section 2. Section 3 contains basic results related to the weighted composition operators from a growth space into a lattice. In particular, we prove Theorem 1.2 and we characterize q -Carleson measures for $\mathcal{F}_\alpha^\infty(\mathbb{C}^d)$, $d \geq 1$, and other growth spaces. Corollaries 1.3 and 1.4 are obtained in Section 4. We prove related corollaries for certain combinations of T_g and C_φ defined on the growth Fock spaces. Also, we compare particular cases of the corollaries obtained and analogous results from [10, 12].

2. AUXILIARY RESULTS

2.1. Approximation by entire maps.

Theorem 2.1 ([1, Theorem 3.5]). *Let $\omega : [0, +\infty) \rightarrow (0, +\infty)$ be a weight function. Then ω is essential if and only if there exists $n \in \mathbb{N}$ and a holomorphic map $f : \mathbb{C}^d \rightarrow \mathbb{C}^n$ such that $|f_1| + \dots + |f_n| \asymp \omega$.*

To work with explicit weight functions ω , we need a sufficient condition for being essential. So, consider the logarithmic transform

$$\Phi_\omega(x) = \log \omega(e^x), \quad -\infty < x < +\infty.$$

If Φ_ω is a convex function, then ω is called log-convex. Observe that the property of being essential for ω depends only on the behavior of Φ_ω at $+\infty$. In fact, Theorem 2.1 implies that equivalence to a log-convex weight function is necessary for being essential. The following sufficient condition is somewhat more stringent.

Theorem 2.2 ([1, paragraph after Example 1]). *Let $\omega : [0, +\infty) \rightarrow (0, +\infty)$ be a log-convex and C^2 -smooth weight function. Assume that*

$$(2.1) \quad \liminf_{x \rightarrow +\infty} \Phi_\omega''(x) > 0.$$

Then ω is essential.

2.2. Characterizations of the Fock spaces.

Theorem 2.3 (see [9, 15]). *Let $\alpha > 0$, $0 < p \leq \infty$ and let $f \in \mathcal{H}ol(\mathbb{C}^d)$, $d \geq 1$. Then $f \in \mathcal{F}_\alpha^p(\mathbb{C}^d)$ if and only if*

$$\frac{|\mathcal{R}f(z)|}{(1+|z|)^2} e^{-\frac{\alpha}{2}|z|^2} \in L^p(\mathbb{C}^d, \nu_d).$$

3. BASIC RESULTS

3.1. Weighted composition operators.

Proof of Theorem 1.2. First, assume that $f \in \mathcal{A}^\omega(\mathbb{C}^d)$ and (1.1) holds, that is, $|g(z)|\tilde{\omega}(|\varphi(z)|) \in \mathcal{L}(\mathbb{C}^d)$. Since $\mathcal{A}^\omega(\mathbb{C}^d)$ isometrically equals $\mathcal{A}^{\tilde{\omega}}(\mathbb{C}^d)$, we have

$$|C_\varphi^g f(z)| \leq \|f\|_{\mathcal{A}^{\tilde{\omega}}(\mathbb{C}^d)} |g(z)|\tilde{\omega}(|\varphi(z)|) \in \mathcal{L}(\mathbb{C}^d).$$

So, $C_\varphi^g f \in \mathcal{L}(\mathbb{C}^d)$ by the definition of a lattice.

Secondly, assume that C_φ maps $\mathcal{A}^\omega(\mathbb{C}^d) = \mathcal{A}^{\tilde{\omega}}(\mathbb{C}^d)$ into $\mathcal{L}(\mathbb{C}^d)$. As indicated in Section 1.2.2, the weight function $\tilde{\omega}$ is essential; thus, applying Theorem 2.1, we obtain functions $f_1, \dots, f_n \in \mathcal{A}^{\tilde{\omega}}(\mathbb{C}^d)$ such that

$$|f_1(z)| + \dots + |f_n(z)| \geq \tilde{\omega}(|z|), \quad z \in \mathbb{C}^d.$$

Hence, for $z \in \mathbb{C}^d$, we have

$$|g(z)|\tilde{\omega}(|\varphi(z)|) \leq \sum_{j=1}^n |g(z)||f_j(\varphi(z))| = \sum_{j=1}^n |C_\varphi^g f_j(z)| \in \mathcal{L}(\mathbb{C}^d).$$

Therefore, (1.1) holds by the definition of a lattice. \square

3.2. Carleson measures. Let $0 < q < \infty$ and let $X \subset \mathcal{H}ol(\mathbb{C}^d)$ be a linear space. A positive Borel measure μ on \mathbb{C}^d is called q -Carleson for X if $X \subset L^q(\mathbb{C}^d, \mu)$. Applying Theorem 1.2 with $\mathcal{L}(\mathbb{C}^d) = L^q(\mathbb{C}^d, \mu)$, $g \equiv 1$ and $\varphi(z) \equiv z$, we obtain the following result.

Corollary 3.1. *Suppose that $\omega : [0, +\infty) \rightarrow (0, +\infty)$ is a weight function, μ is a positive Borel measure on \mathbb{C}^d , and $0 < q < \infty$. Then μ is a q -Carleson measure for $\mathcal{A}^\omega(\mathbb{C}^d)$ if and only if*

$$(3.1) \quad \int_{\mathbb{C}^d} \tilde{\omega}^q(|z|) d\mu(z) < \infty.$$

3.2.1. Fock–Sobolev Carleson measures. The Fock type spaces $\mathcal{F}_{1,\beta}^p(\mathbb{C}^d)$, $\beta \in \mathbb{R}$, $0 < p \leq \infty$, are introduced in [4]. In particular, $\mathcal{F}_{1,\beta}^\infty(\mathbb{C}^d) = \mathcal{A}^{\omega_{1,\beta}}(\mathbb{C}^d)$, where $\omega_{1,\beta}(t) = (1+t)^\beta e^{\frac{1}{2}t^2}$, $0 \leq t < +\infty$. As shown in [4], any Fock–Sobolev space $\mathcal{F}_{1,\beta,s}^p(\mathbb{C}^d)$ of fractional order $s \in \mathbb{R}$ coincides with $\mathcal{F}_{1,\beta-s}^p(\mathbb{C}^d)$, $0 < p \leq \infty$, $\beta \in \mathbb{R}$. This fact allows to extend most of the results of the present paper to $\mathcal{F}_{1,\beta}^\infty(\mathbb{C}^d)$, $\beta \in \mathbb{R}$. We restrict our attention to characterizations of q -Carleson measures for $\mathcal{F}_{1,\beta}^p(\mathbb{C}^d)$. For $0 < p < \infty$, the problem is solved in [4]. So, we consider the case where $p = \infty$.

Corollary 3.2. *Suppose that $d \geq 1$, $\beta \in \mathbb{R}$, $0 < q < \infty$ and μ is a positive Borel measure on \mathbb{C}^d . Then μ is a q -Carleson measure for $\mathcal{F}_{1,\beta}^\infty(\mathbb{C}^d)$ if and only if*

$$\int_{\mathbb{C}^d} (1 + |z|^\beta)^q e^{\frac{\beta}{2}|z|^2} d\mu(z) < \infty.$$

Proof. The logarithmic transform $\Phi_{\omega_{1,\beta}}(x) = \beta \log(1 + e^x) + \frac{1}{2}e^{2x}$, $-\infty < x < +\infty$, clearly satisfies condition (2.1). So, it suffices to apply Theorem 2.2 and Corollary 3.1. \square

3.2.2. Carleson measures for $\mathcal{F}_\alpha^\infty(\mathbb{C}^d)$. We have $\mathcal{F}_\alpha^\infty(\mathbb{C}^d) = \mathcal{A}^{\omega_\alpha}(\mathbb{C}^d)$ with $\omega_\alpha(t) = e^{\frac{\alpha}{2}t^2}$, $\alpha > 0$, $0 \leq t < \infty$. So, Theorem 2.2 and Corollary 3.1 guarantee that μ is a q -Carleson measure for $\mathcal{F}_\alpha^\infty(\mathbb{C}^d)$, $d \geq 1$, if and only if

$$\int_{\mathbb{C}^d} e^{\frac{\alpha q}{2}|z|^2} d\mu(z) < \infty.$$

For $d = 1$, the above condition was obtained in [10] by a different method.

3.3. Volterra type operators. Recall that the Volterra type operator V_g^φ is defined as

$$(V_g^\varphi f)(z) = \int_0^1 f(\varphi(tz)) \frac{\mathcal{R}g(tz)}{t} dt, \quad f \in \mathcal{H}ol(\mathbb{C}^d), \quad z \in \mathbb{C}^d,$$

where $g \in \mathcal{H}ol(\mathbb{C}^d)$ and $\varphi : \mathbb{C}^d \rightarrow \mathbb{C}^d$ is a holomorphic map. Direct calculations guarantee that

$$(3.2) \quad \mathcal{R}V_g^\varphi f(z) = f(\varphi(z))\mathcal{R}g(z), \quad z \in \mathbb{C}^d,$$

for all $f, g \in \mathcal{H}ol(\mathbb{C}^d)$; see [8].

Corollary 3.3. *Suppose that $\omega : [0, +\infty) \rightarrow (0, +\infty)$ is a weight function, $g \in \mathcal{H}ol(\mathbb{C}^d)$, $\varphi : \mathbb{C}^d \rightarrow \mathbb{C}^d$ is a holomorphic map, and $\mathcal{L}(\mathbb{C}^d)$ is a lattice. Then $\mathcal{R}V_g^\varphi$ maps $\mathcal{A}^\omega(\mathbb{C}^d)$ into $\mathcal{L}(\mathbb{C}^d)$ if and only if $|\mathcal{R}g(z)|\tilde{\omega}(|\varphi(z)|) \in \mathcal{L}(\mathbb{C}^d)$.*

Proof. It suffices to apply (3.2) and Theorem 1.2. \square

3.4. Compact operators. Characterizations of the compact operators considered in Theorem 1.2 and its corollaries depend on the structure of the corresponding lattice $\mathcal{L}(\mathbb{C}^d)$. If $\mathcal{L}(\mathbb{C}^d)$ is an L^∞ space, then, as indicated in Section 1.1, the little-oh versions of the boundedness conditions describe the corresponding compact operators. If $\mathcal{L}(\mathbb{C}^d) = L^q(\mathbb{C}^d, \mu)$ with $0 < q < \infty$, then the bounded operators under consideration are automatically compact. For example, given $0 < q < \infty$ and a positive Borel measure μ on \mathbb{C}^d , standard arguments show that the identity operator $\text{Id} : \mathcal{A}^\omega(\mathbb{C}^d) \rightarrow L^q(\mathbb{C}^d, \mu)$ is compact if and only if (3.1) holds. So, in what follows, we restrict our attention to the boundedness conditions.

4. OPERATORS ON THE GROWTH FOCK SPACES

4.1. Corollary 1.3 and related results.

Proof of Corollary 1.3. By Theorem 2.3, property (i) from Corollary 1.3 holds if and only if

$$\frac{(\mathcal{R}V_g^\varphi f)(z)}{(1+|z|)^2} e^{-\frac{\alpha}{2}|z|^2} \in L^q(\mathbb{C}^d, \nu_d)$$

for all $f \in \mathcal{A}^\omega(\mathbb{C}^d) = \mathcal{A}^{\tilde{\omega}}(\mathbb{C}^d)$. Let $\mathcal{L}(\mathbb{C}^d)$ consist of those measurable $F : \mathbb{C}^d \rightarrow \mathbb{C}$ for which

$$\frac{|F(z)|}{(1+|z|)^2} e^{-\frac{\alpha}{2}|z|^2} \in L^q(\mathbb{C}^d, \nu_d).$$

Corollary 3.3 guarantees that (i) is equivalent to the following property:

$$(4.1) \quad \frac{|\mathcal{R}g(z)|\tilde{\omega}(|\varphi(z)|)}{(1+|z|)^2} e^{-\frac{\alpha}{2}|z|^2} \in L^q(\mathbb{C}^d, \nu_d),$$

as required. \square

4.1.1. Berezin type characterizations. The normalized kernel function $k_{w,\alpha}$ for the Hilbert space $\mathcal{F}_\alpha^2(\mathbb{C}^d)$ is given by the identity

$$(4.2) \quad k_{w,\alpha}(z) = e^{\alpha\langle z,w \rangle - \alpha|w|^2/2}, \quad z, w \in \mathbb{C}^d.$$

To study C_φ^g , V_g^φ and related operators on Fock type spaces, Ueki [14] and Mengestie [10, 11, 12] introduce Berezin type integral transforms with the help of $k_{w,\alpha}(z)$. For example, the bounded (compact) Volterra type operators $V_g^\varphi : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ are characterized by the following theorem.

Theorem 4.1 ([10, Theorem 2.3]). *Assume that $g, \varphi \in \mathcal{H}ol(\mathbb{C})$, $\alpha > 0$ and $0 < q < \infty$. Then the following properties are equivalent:*

- (i) $V_g^\varphi : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ is a bounded operator;
- (ii) $V_g^\varphi : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ is a compact operator;
- (iii)

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \frac{|k_{w,\alpha}(\varphi(z))g'(z)|^q}{(1+|z|)^q} e^{-\frac{\alpha q}{2}|z|^2} d\nu_1(z) d\nu_1(w) < \infty.$$

4.1.2. *Comparison of Corollary 1.3 and Theorem 4.1.* We have $\mathcal{F}_\alpha^\infty(\mathbb{C}^d) = \mathcal{A}^{\omega_\alpha}(\mathbb{C}^d)$ with $\omega_\alpha(t) = e^{\frac{\alpha}{2}t^2}$, $\alpha > 0$, $0 \leq t < +\infty$. As indicated in Section 3.2, Theorem 2.2 guarantees that $\tilde{\omega}_\alpha \asymp \omega_\alpha$. Hence, (4.1) with $\omega = \omega_\alpha$ reads as

$$\frac{|\mathcal{R}g(z)|}{(1+|z|)^2} e^{\frac{\alpha}{2}(|\varphi(z)|^2 - |z|^2)} \in L^q(\mathbb{C}^d, \nu_d).$$

If $d = 1$, then $\mathcal{R}g(z) = zg'(z)$, $z \in \mathbb{C}$. Thus, the following condition is equivalent to properties (i–iii) from Theorem 4.1:

$$(4.3) \quad \frac{|g'(z)|}{1+|z|} e^{\frac{\alpha}{2}(|\varphi(z)|^2 - |z|^2)} \in L^q(\mathbb{C}, \nu_1).$$

If $g(z)$ is a constant, then $V_g^\varphi = 0$. If $g(z)$ is not a constant, then (4.3) allows to make certain explicit conclusions about those symbols g and φ for which $V_g^\varphi : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$, $0 < q < \infty$, is bounded or, equivalently, compact.

Corollary 4.2. *Let $g, \varphi \in \text{Hol}(\mathbb{C})$, $\alpha > 0$ and $0 < q < \infty$.*

- (i) *Assume that $g(z)$ grows as a power function of degree at least 2 and $V_g^\varphi : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ is bounded. Then $\varphi(z) = \beta z + \gamma$ with $|\beta| < 1$.*
- (ii) *Assume that $g(z) = az + b$, $a \neq 0$, and $0 < q \leq 2$. Then $V_g^\varphi : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ is bounded (compact) if and only if $\varphi(z) = \beta z + \gamma$ with $|\beta| < 1$.*
- (iii) *Assume that $g(z) = az + b$, $a \neq 0$, and $2 < q < \infty$. Then $V_g^\varphi : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ is bounded (compact) if and only if $\varphi(z) = \beta z + \gamma$ with $|\beta| \leq 1$.*

4.2. Corollary 1.4 and related results.

Proof of Corollary 1.4. Observe that $C_{\varphi,g}f(z) = \int_0^{\varphi(z)} f(\xi)g'(\xi)d\xi$, $z \in \mathbb{C}$. So, Theorem 2.3 guarantees that $C_{\varphi,g}$ maps $\mathcal{A}^\omega(\mathbb{C})$ into $\mathcal{F}_\alpha^q(\mathbb{C})$ if and only if

$$\frac{|f(\varphi(z))g'(\varphi(z))\varphi'(z)|}{1+|z|} e^{-\frac{\alpha}{2}|z|^2} \in L^q(\mathbb{C}, \nu_1) \quad \text{for all } f \in \mathcal{A}^\omega(\mathbb{C}).$$

Since $\tilde{\omega}$ is essential, the above condition is equivalent to the following one:

$$(4.4) \quad \frac{|g'(\varphi(z))\varphi'(z)|}{1+|z|} \tilde{\omega}(|\varphi(z)|) e^{-\frac{\alpha}{2}|z|^2} \in L^q(\mathbb{C}, \nu_1)$$

by Theorem 1.2. The proof of Corollary 1.4 is finished. \square

4.2.1. *Berezin type characterizations.* By [10, Theorem 2.4], the operator $C_{\varphi,g} : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$, $0 < q < \infty$, is bounded (compact) if and only if

$$(4.5) \quad \int_{\mathbb{C}} \int_{\mathbb{C}} \frac{|k_{w,\alpha}(\varphi(z))g'(\varphi(z))\varphi'(z)|^q}{((1+|z|)e^{\frac{\alpha}{2}|z|^2})^q} d\nu_1(z) d\nu_1(w) < \infty,$$

where $k_{w,\alpha}(z)$ is defined by (4.2).

4.2.2. *Comparison of conditions (4.4) and (4.5).* For $\omega_\alpha(t) = e^{\frac{\alpha}{2}t^2}$, $\alpha > 0$, $0 \leq t < \infty$, condition (4.4) rewrites as

$$(4.6) \quad \frac{|g'(\varphi(z))\varphi'(z)|}{1+|z|} e^{\frac{\alpha}{2}(|\varphi(z)|^2 - |z|^2)} \in L^q(\mathbb{C}, \nu_1).$$

In particular, (4.5) and (4.6) are equivalent. If g is a constant, then $T_g^\varphi = 0$. Using (4.6), we also make the following explicit conclusions.

Corollary 4.3. *Let $g, \varphi \in \text{Hol}(\mathbb{C})$, $\alpha > 0$ and $0 < q < \infty$.*

- (i) Assume that $g(z)$ is not a constant and $T_g^\varphi : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ is bounded. Then $\varphi(z) = \beta z + \gamma$ with $|\beta| < 1$.
- (ii) Assume that $g(z) = az + b$, $a \neq 0$. Then $T_g^\varphi : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ is bounded (compact) if and only if $\varphi(z) = \beta z + \gamma$ with $|\beta| < 1$.

4.3. Volterra companion integral operators. Let $g \in \mathcal{Hol}(\mathbb{C})$. Then the operator T_g , companion to V_g , is defined by

$$T_g f(z) = \int_0^z f'(\xi) g(\xi) d\xi, \quad \xi \in \mathbb{C}.$$

Companions of V_g^φ and $C_{\varphi,g}$ are defined as

$$T_g^\varphi f(z) = \int_0^z f'(\varphi(\xi)) g(\xi) d\xi, \quad K_{\varphi,g} f(z) = \int_0^{\varphi(z)} f'(\xi) g(\xi) d\xi, \quad z \in \mathbb{C}.$$

The bounded and compact operators $T_g^\varphi, K_{\varphi,g} : \mathcal{F}_\alpha^p(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ are characterized in [12] for all $0 < p, q \leq +\infty$ in terms of Berezin type transforms. We apply Theorem 2.1 to give more explicit descriptions for $p = \infty$.

Corollary 4.4. Assume that $g, \varphi \in \mathcal{Hol}(\mathbb{C})$, $\alpha > 0$ and $0 < q \leq \infty$. Then the following properties are equivalent:

- (i) $T_g^\varphi : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ is a bounded operator;
- (ii)

$$\frac{1 + |\varphi(z)|}{1 + |z|} |g(z)| e^{\frac{\alpha}{2}(|\varphi(z)|^2 - |z|^2)} \in L^q(\mathbb{C}, \nu_1).$$

Proof. First, Theorem 2.3 guarantees that property (i) holds if and only if

$$(4.7) \quad \frac{|f'(\varphi(z))g(z)|}{1 + |z|} e^{-\frac{\alpha}{2}|z|^2} \in L^q(\mathbb{C}, \nu_1) \quad \text{for all } f \in \mathcal{F}_\alpha^\infty(\mathbb{C}).$$

If $f \in \mathcal{F}_\alpha^\infty(\mathbb{C})$, then $|f'(\xi)| \leq (1 + |\xi|)e^{\frac{\alpha}{2}|\xi|^2}$, $\xi \in \mathbb{C}$, by Theorem 2.3 with $p = \infty$. Therefore, (ii) implies (i).

Secondly, assume that (i) holds. Theorem 2.2 is applicable to the weight function $\omega_{\alpha,1}(t) = (1 + t)e^{\frac{\alpha}{2}t^2}$, $0 \leq t < \infty$. Hence, Theorem 2.1 provides functions $h_1, \dots, h_n \in \mathcal{Hol}(\mathbb{C})$ such that

$$(4.8) \quad |h_1(z)| + \dots + |h_n(z)| \asymp \omega_{\alpha,1}(|z|), \quad z \in \mathbb{C}.$$

Put

$$f_j(z) = \int_0^z h_j(\xi) d\xi, \quad z \in \mathbb{C}.$$

Then $f_j' = h_j$ and $f_j \in \mathcal{F}_\alpha^\infty(\mathbb{C})$, $j = 1, \dots, n$, by (4.8) and Theorem 2.3 with $p = \infty$. So, putting $f = f_j$, $j = 1, \dots, n$, in (4.7), taking the sum and using (4.8), we conclude that (i) implies (ii). \square

Corollary 4.5. Let $\varphi, g \in \mathcal{Hol}(\mathbb{C})$, $\alpha > 0$ and $0 < q < \infty$. Assume that $g(z) \not\equiv 0$ and $K_{\varphi,g} : \mathcal{F}_\alpha^\infty(\mathbb{C}) \rightarrow \mathcal{F}_\alpha^q(\mathbb{C})$ is bounded. Then $\varphi(z) = \beta z + \gamma$ with $|\beta| < 1$.

The proof of the following explicit description for $K_{\varphi,g}$ is similar to that of Corollary 4.4.

Corollary 4.6. *Assume that $\varphi, g \in \text{Hol}(\mathbb{C})$, $\alpha > 0$ and $0 < q \leq \infty$. Then $K_{\varphi, g} : \mathcal{F}_{\alpha}^{\infty}(\mathbb{C}) \rightarrow \mathcal{F}_{\alpha}^q(\mathbb{C})$ is a bounded operator if and only if*

$$\frac{1 + |\varphi(z)|}{1 + |z|} |g(\varphi(z))\varphi'(z)| e^{\frac{\alpha}{2}(|\varphi(z)|^2 - |z|^2)} \in L^q(\mathbb{C}, \nu_1).$$

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